

# Classical Geometry and Target Space Duality\*

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## Abstract

A new formulation for a “restricted” type of target space duality in classical two dimensional nonlinear sigma models is presented. The main idea is summarized by the analogy: euclidean geometry is to riemannian geometry as toroidal target space duality is to “restricted” target space duality. The target space is not required to possess symmetry. These lectures only discuss the local theory. The restricted target space duality problem is identified with an interesting problem in classical differential geometry.

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# 1 Introduction

Target space duality is a remarkable phenomenon where different 2-dimensional non-linear sigma models are physically equivalent. A key reason for interest in this subject is that duality often turns a strong coupling problem into an equivalent weak coupling one thus transforming an intractable problem into a manageable one. There are two questions which immediately come to mind. The difficult one is: given a sigma model does there exist a dual model? A more accessible one is: when are two sigma models dual to each other? In these lectures we will attempt to address the latter question within the framework of classical hamiltonian mechanics.

Basically, two sigma models are target space duals of each other if there exists a canonical transformation between the phase spaces which preserves the respective hamiltonians [1, 2, 3, 4]. The existence of such a map is a difficult question to determine because the phase spaces are infinite dimensional. To make progress, we look for guidance in explicit examples. The case of toroidal target spaces suggests a promising approach. The duality transformation between the infinite dimensional phase spaces, in the case of toroidal target spaces, may be viewed as being induced by a special map between some finite dimensional bundles over the target spaces. In these lectures we address whether a similar phenomenon can arise between more general targets. We will see that it is possible to look for a special type of “restricted” target space duality which leads to an intriguing problem in classical differential geometry. There is a known duality transformation which arises when one of the target spaces is a simple Lie group. The explicit form of the generating function suggests a generalization to more general manifolds. This plays a pivotal role in our formulation.

There is a lot known about target space duality when the target is a torus. The excellent review article of Giveon, Porrati and Rabinovici [5] discusses the physics arising from toroidal targets in great detail. Additionally, there are roughly 300 references to the literature in this review which allow the reader to explore the historical development of the subject. We will use [5] as our unique reference on toroidal target spaces. The more recent phenomenon of “non-abelian” duality goes back the work presented in [6] but can actually be traced back to a much older paper [7]. The notion that duality can be formulated as canonical transformation goes back to [1] even though in the 1970’s much work was done in statistical mechanics on the study of abelian duality in lattice systems from the partition function viewpoint. The explicit construction of a canonical transformation including the generating function for the target space  $SU(2)$  is due to [2].

euclidean geometry	<i>is to</i>	riemannian geometry
	as	
toroidal target space duality	<i>is to</i>	“restricted” target space duality

Table 1: The key analogy.

The approach taken in these lectures is different from the traditional approaches to duality presented in the literature (see for example the discussions in [6, 8, 2, 9, 10, 11]). In all these approaches one has explicitly symmetries in the target space which play a central role in the discussion of duality. The duality transformation in these theories with symmetry is some type of generalized Fourier transform. I was looking for a formulation which did not depend on the existence of symmetries. I wanted something which might be applicable to mirror symmetry [12]. The key analogy to keep in mind while reading these lectures is presented in Table 1. Mostly I will present ideas and concepts rather than detailed mathematical formulas. The derivation of explicit formulas requires a discussion of the theory of  $G$ -structures, a discussion of the differential forms version of the Frobenius theorem, and a presentation of Cartan’s equivalence method [13]. These topics are outside the scope of these lectures and will be presented in Part I of [14]. The ideas presented here can be generalized by weakening the requirement that the canonical transformation be induced by a finite dimensional map. This requires the full machinery of Cartan-Kahler theory [15] and will be presented in Part II of [14]. These ideas can be extended to complex manifolds as discussed in [14].

## 2 Preliminaries

The classical nonlinear sigma model is defined by a map  $x$  from a lorentzian world sheet  $\Sigma$  to a target manifold  $M$  and some additional geometric data which specifies the lagrangian. In this article we will take the world sheet  $\Sigma$  to be either  $\mathbf{R} \times \mathbf{R}$  or  $\mathbf{R} \times S^1$ . The first factor is time and the second factor is space. Local coordinates on  $\Sigma$  will be denoted by  $(\tau, \sigma)$ . The target space  $M$  is endowed with a metric tensor  $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$  and a 2-form  $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ . The lagrangian density for this model is

$$\mathcal{L} = \frac{1}{2} \eta^{ab} g_{\mu\nu}(x) \partial_a x^\mu \partial_b x^\nu - \frac{1}{2} \epsilon^{ab} B_{\mu\nu}(x) \partial_a x^\mu \partial_b x^\nu , \quad (1)$$

where  $\eta$  is the two dimensional lorentzian metric on the world sheet. A good way of denoting the sigma model is to use the notation  $(M, ds^2, B)$  which incorporates all the relevant geometrical data. The canonically conjugate momenta are given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}(x) \dot{x}^\nu + B_{\mu\nu}(x) \frac{\partial x^\nu}{\partial \sigma} . \quad (2)$$

Local coordinates in phase space may be taken to be  $(x(\sigma), \pi(\sigma))$ . If the spatial part of  $\Sigma$  is a circle then  $(x(\sigma), \pi(\sigma))$  is a loop in  $T^*M$ , the cotangent bundle of  $M$ . If the spatial part of  $\Sigma$  is  $\mathbf{R}$  then  $(x(\sigma), \pi(\sigma))$  is a path in  $T^*M$ . The symplectic form on the phase space is given by

$$\int \delta \pi_\mu(\sigma) \wedge \delta x^\mu(\sigma) d\sigma , \quad (3)$$

where  $\delta$  is the differential on the phase space. With this symplectic structure we see that the basic Poisson bracket is given by

$$\{x^\mu(\sigma), \pi_\nu(\sigma')\}_{\text{PB}} = \delta^\mu{}_\nu \delta(\sigma - \sigma') . \quad (4)$$

The hamiltonian density and the worldsheet momentum density are respectively given by

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu}(x) \left( \pi_\mu - B_{\mu\kappa} \frac{dx^\kappa}{d\sigma} \right) \left( \pi_\nu - B_{\nu\lambda} \frac{dx^\lambda}{d\sigma} \right) + \frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} , \quad (5)$$

$$\mathcal{P} = \pi_\mu \frac{dx^\mu}{d\sigma} . \quad (6)$$

Target space duality is the phenomenon that a nonlinear sigma model  $(M, ds^2, B)$  is *equivalent* to a different nonlinear sigma model  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$ . We need to define *equivalent*. For the moment we will make a preliminary definition which will be modified later.

**Definition 7 (Preliminary)** *Two sigma models  $(M, ds^2, B)$  and  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$  are said to be target space dual to each other if there exists a canonical transformation from the phase space of  $(M, ds^2, B)$  to the phase space of  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$  which maps the hamiltonian  $\mathcal{H}$  of the first model to the hamiltonian  $\widetilde{\mathcal{H}}$  of the second model.*

Later we will see that there are some important domain and range issues which must be addressed to have a good definition of target space duality.

### 3 Examples

### 3.1 Circular target space

As an example of the above we consider the case where the target space is a circle of radius  $R$  (for a more detailed discussion look in [5]). The coordinate  $x$  on the circle has period  $2\pi R$ . The hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{dx}{d\sigma}\right)^2. \quad (8)$$

We can define a *formal* canonical transformation by the ordinary differential equations

$$\frac{d\tilde{x}}{d\sigma} = \pi(\sigma), \quad (9)$$

$$\tilde{\pi} = \frac{dx}{d\sigma}. \quad (10)$$

It is clear that the hamiltonian is preserved by this map. The new hamiltonian tells us that the new target space is either  $\mathbf{R}$  or  $S^1$ .

The transformation above is formal because of certain domain and range issues. First we note that there is an  $S^1$  action on our circle given by translating  $x$ . Noether's theorem leads to the conserved target space momentum  $P = \int \pi(\sigma)d\sigma$  associated with translations in the target space. For the moment let us assume the world sheet is  $\Sigma = \mathbf{R} \times S^1$ . In this case, the difference  $x(2\pi) - x(0)$  is quantized in units of  $2\pi R w$  where the winding number  $w$  is an integer. Consequently, phase space divides into sectors labeled by  $[w, p_T]$  where  $w$  is the winding number and  $p_T$  is the total target space momentum. Since both these quantities are conserved, the subspace labeled by  $[w, p_T]$  will be invariant under the hamiltonian flow. This observation has important consequences when we examine the duality transformation in more detail. Integrating equations (9) and (10) we see that:

$$\tilde{x}(2\pi) - \tilde{x}(0) = \int_{S^1} \pi(\sigma)d\sigma, \quad (11)$$

$$\int_{S^1} \tilde{\pi}(\sigma)d\sigma = x(2\pi) - x(0). \quad (12)$$

The above indicates that the dual target manifold should be a circle of radius  $\tilde{R}$  and we should have relations  $2\pi\tilde{R}\tilde{w} = p_T$  and  $2\pi R w = \tilde{p}_T$ . Since the winding number  $\tilde{w}$  must be an integer we see that  $p_T$  must be “classically quantized” and likewise  $\tilde{p}_T$ . At the classical level we have the following: given two radii  $R$  and  $\tilde{R}$ , the sigma model with radius  $R$  on the reduced phase space characterized by  $[w, 2\pi\tilde{R}\tilde{w}]$  is dual to the sigma model on a circle of radius  $\tilde{R}$  on the reduced phase space characterized by  $[\tilde{w}, 2\pi R w]$ . Note that some type of “pre-quantization” has taken place in trying to define duality.

There is no “good” map from the full phase space of the nonlinear sigma model on a circle of radius  $R$  to the nonlinear sigma model on a circle of radius  $\tilde{R}$ . Under quantization we observe that the momentum  $P$  must be quantized in units of  $1/R$ , likewise, the momentum  $\tilde{P}$  must be quantized in units of  $1/\tilde{R}$ . Incorporating this we see that there is now a relation  $2\pi R\tilde{R} = 1$  between  $R$  and  $\tilde{R}$ . Note that if we take  $\Sigma = \mathbf{R} \times \mathbf{R}$  then the winding number is not defined and the domain and range issues do not appear. The lesson learned in this example is that there are delicate domain and range issues which must be understood if one wants to be mathematically precise.

Preliminary definition (7) must be expanded to include domain and range information. A better definition of classical duality would be

**Definition 13** *Two sigma models  $(M, ds^2, B)$  and  $(\tilde{M}, \tilde{ds}^2, \tilde{B})$  are said to be target space dual to each other if there exists a canonical transformation from a reduced phase space of  $(M, ds^2, B)$  to a reduced phase space of  $(\tilde{M}, \tilde{ds}^2, \tilde{B})$  which maps the hamiltonian  $\mathcal{H}$  of the first model to the hamiltonian  $\tilde{\mathcal{H}}$  of the second model. The reduced phase spaces must be invariant under the respective hamiltonian flow.*

### 3.2 Toroidal target spaces

In this example we choose the target space to be an  $n$ -dimensional torus  $\mathbf{T}^n$  (for a more detailed discussion look in [5]). The metric  $ds^2$  and the 2-form  $B$  are chosen to be constant. The basic Poisson brackets for this model are given by equation (4). By differentiating they may be written as

$$\left\{ \frac{dx^\mu}{d\sigma}(\sigma), \pi_\nu(\sigma') \right\}_{\text{PB}} = \delta^\mu_\nu \delta'(\sigma - \sigma') . \quad (14)$$

If we now put  $dx/d\sigma$  and  $\pi$  into a  $2n$ -vector

$$z(\sigma) = \begin{pmatrix} dx/d\sigma \\ \pi \end{pmatrix} \quad (15)$$

then the Poisson brackets may be written as

$$\{z^A(\sigma), z^B(\sigma')\}_{\text{PB}} = Q^{AB} \delta'(\sigma - \sigma') , \quad (16)$$

where

$$Q = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} . \quad (17)$$

The indices  $A$  and  $B$  range over  $\{1, \dots, 2n\}$ . We see that a constant linear transformation  $T : z \mapsto Tz$  will preserve the symplectic structure if  $T$  is in the pseudo-orthogonal

group  $O_Q(2n)$  consisting of linear transformations which preserve the quadratic form  $Q$ . This group is isomorphic to  $O(n, n)$ . Since the hamiltonian is a quadratic form in the  $z$ 's with constant coefficients, we see that a  $T \in O_Q(2n)$  leads to a new sigma model hamiltonian with constant coefficients. A similar phenomenon will be studied in detail in a more general setting later. For the moment we make a few observations. Since the hamiltonian is a positive definite quadratic form in  $\{dx/d\sigma, \pi\}$ , the linear transformations in  $O_Q(2n)$  that preserve the quadratic form belong to a certain maximal compact subgroup  $K$  which is isomorphic to  $O(n) \times O(n)$ . The dimension of the coset space  $O_Q(2n)/K$  is  $n^2$  which is precisely the total number of independent components in the metric  $ds^2$  and the 2-form  $B$ . In fact the coset space  $O_Q(2n)/K$  parameterizes the space of nonlinear sigma model hamiltonians with constant coefficients. Actually one has to be careful in the quantum theory. One can show that transformations in the subgroup  $O(n, n; \mathbf{Z})$  lead to equivalent hamiltonians.

In this report we are interested in local conditions which necessarily guarantee the existence of a restricted type of target space duality. Because we are only interested in local issues we will generally take  $\Sigma = \mathbf{R} \times \mathbf{R}$ .

### 3.3 Toroidal target spaces revisited

We revisit toroidal target spaces and adopt a different viewpoint which will generalize to generic targets. There is a very interesting structure which arises in the models we have been studying. Introduce a new space isomorphic to  $\mathbf{R}^n$  with coordinates  $p$ . Define  $p(\sigma)$  by

$$\frac{dp}{d\sigma} = \pi(\sigma) . \quad (18)$$

Note that there is an ambiguity in the definition of  $p$  due to the constant of integration. Instead of working in phase space  $(x(\sigma), \pi(\sigma))$  we can work in a space with coordinates  $(x(\sigma), p(\sigma))$ . In terms of these new variables the symplectic form may be written as

$$\int \delta x^\mu(\sigma) \wedge \frac{d}{d\sigma} \delta p_\mu(\sigma) d\sigma = \frac{1}{2} \int d\sigma \begin{pmatrix} \delta x & \delta p \end{pmatrix} \wedge \frac{d}{d\sigma} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta p \end{pmatrix} . \quad (19)$$

This symplectic form is degenerate. We have also discarded a surface term. We will ignore these technical issues. Note that the matrix  $Q$  enters into this formulation of the symplectic form. Also, a constant coefficient linear transformation  $T \in O_Q(2n)$  acting on  $(x, p)$  space will preserve this symplectic form. Using the variables  $(x, p)$  we can integrate equations (9), (10), and obtain

$$\tilde{x}(\sigma) = p(\sigma) + a , \quad (20)$$

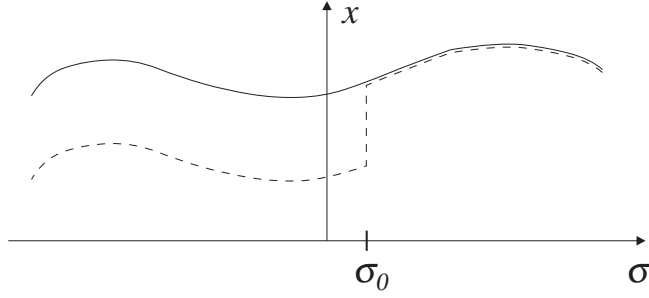


Figure 1: The action of  $\exp[2\pi i R p(\sigma_0)]$  takes a state with expectation value  $x_{\text{cl}}(\sigma)$  (the solid curve) to a state with expectation value  $x_{\text{cl,new}}(\sigma)$  (the dashed curve).

$$\tilde{p}(\sigma) = x(\sigma) + b, \quad (21)$$

where  $a$  and  $b$  are constants. In passing we mention that if  $\Sigma = \mathbf{R} \times S^1$  and if we naively quantize then there will be appropriate periodicity requirements on  $(x, p)$  and  $(\tilde{x}, \tilde{p})$ . The map  $(x, p) \mapsto (\tilde{x} = p + a, \tilde{p} = x + b)$  between the toroidal target spaces  $\mathbf{T}^{2n}$  induces transformations (20) and (21) on the space of paths.

The question which will occupy our attention throughout this report is whether it is possible to formulate target space duality as an induced map between the infinite dimensional phase spaces arising from an ordinary map from a  $2n$ -manifold to another  $2n$ -manifold.

The variables  $p$  have an important physical significance. They are the variables which describe the “solitons” in toroidal models. It is best to take a semi-classical viewpoint. By the canonical commutation relations we have that

$$\pi_\mu(\sigma) = -i \frac{\delta}{\delta x^\mu(\sigma)}. \quad (22)$$

Therefore we have

$$\exp[i\alpha^\mu p_\mu(\sigma)] = \exp\left(\alpha^\mu \int_{-\infty}^{\sigma} d\sigma' \frac{\delta}{\delta x^\mu(\sigma')}\right). \quad (23)$$

This equation tells us that  $\exp[i\alpha \cdot p(\sigma)]$  is the operator which creates a “kink” with jump of size  $\alpha$ . To see this let us assume the target space is a circle of radius  $R$  and consider a state with the property that  $\langle x(\sigma) \rangle = x_{\text{cl}}(\sigma)$ . The action of the operator  $\exp[2\pi i R p(\sigma_0)]$  on the state leads to a new state with expectation value  $x_{\text{cl,new}}(\sigma)$  which contains a  $2\pi R$  kink at  $\sigma_0$  as in Figure 1.

In the toroidal models we see that the space with coordinates  $(x, p)$  is a space which describes both the “particles”, *i.e.*, the  $x$ ’s, and the “solitons”, *i.e.*, the  $p$ ’s. The torus



$\mathbf{T}^{2n}$  is determined by a lattice in  $\mathbf{R}^{2n}$ . The group  $O_Q(2n)$  naturally acts on this  $\mathbf{R}^{2n}$ . The lattice will be invariant under  $O(n, n; \mathbf{Z}) \subset O_Q(2n)$ . This action preserves the symplectic form (19).

## 4 Generating functions

The term *generating function* is used in a variety of different contexts in mechanics. Assume we have a hamiltonian system. The equations of motion may be derived as the extremals of the variational principle defined by the action  $I = \int (pdq - H(q, p)dt)$ . Assume we have a second system with canonical coordinates  $(\tilde{q}, \tilde{p})$ , time independent hamiltonian  $\tilde{H}(\tilde{q}, \tilde{p})$  and action  $\tilde{I}$ . The variational principle for  $I$  is equivalent to the variational principle for  $\tilde{I}$  if  $pdq - Hdt$  differs from  $\tilde{p}d\tilde{q} - \tilde{H}dt$  by a total differential:

$$\tilde{p}d\tilde{q} - \tilde{H}dt = pdq - Hdt + dG . \quad (24)$$

If the function  $G$  is time independent then

$$\tilde{p} = \frac{\partial G}{\partial \tilde{q}} , \quad (25)$$

$$p = -\frac{\partial G}{\partial q} , \quad (26)$$

and  $H = \tilde{H}$ . The function  $G(q, \tilde{q})$  is called the *generating function*. A geometrical discussion from the viewpoint of symplectic geometry may be found in [16, 17].

For example, in the simple harmonic oscillator with  $H = \frac{1}{2}(p^2 + q^2)$ , the generating function  $G(q, \tilde{q}) = q\tilde{q}$  leads to the “duality transformation”  $\tilde{p} = q$  and  $p = -\tilde{q}$ .

In a field theoretic context we have that for a circular target space the duality transformation is generated by the functional

$$G[x, \tilde{x}] = \int \tilde{x}(\sigma) \frac{dx}{d\sigma} d\sigma . \quad (27)$$

A simple computation shows that

$$\begin{aligned} \tilde{\pi} &= \frac{\delta G}{\delta \tilde{x}} = \frac{dx}{d\sigma} , \\ \pi &= -\frac{\delta G}{\delta x} = \frac{d\tilde{x}}{d\sigma} . \end{aligned}$$

Note that the integrand is the pullback of a 1-form on the product space  $S^1 \times S^1$  of the two target circles, and that  $G$  is reparametrization invariant.

## 5 Generic target space

### 5.1 Background

We now address whether it is possible to construct some type of theory which addresses duality in a generic sigma model  $(M, ds^2, B)$ . In our construction, the group  $O_Q(2n)$  will appear but in a different manner. The reader is reminded of the analogy presented in Table 1. In these notes we will address a certain “restricted” type of duality which leads to a well-posed mathematical problem. The more general discussion requires the use of the full machinery of exterior differential systems and is beyond the scope of these lectures [14]. Remember that the circular duality transformation may viewed as an induced map on paths arising from an affine map from  $(x, p)$ -space to  $(\tilde{x}, \tilde{p})$ -space. In this section we address whether it is possible to have a similar phenomenon arise when we have a generic target space. We will see that in certain situations duality can arise as an ordinary map from a certain  $2n$ -manifold to another  $2n$ -manifold.

The starting point of our discussion will be generating functions. Assume we have two sigma models  $(M, ds^2, B)$  and  $(\tilde{M}, \tilde{ds}^2, \tilde{B})$ . We would like to know if there is duality transformation between these models. Let us postulate that the generating function has the form

$$G[x, \tilde{x}] = \int \left( u_\mu(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\mu}{d\sigma} + v_\mu(x(\sigma), \tilde{x}(\sigma)) \frac{d\tilde{x}^\mu}{d\sigma} \right) d\sigma . \quad (28)$$

This generating function is a generalization to arbitrary manifolds of the one postulated for  $SU(2)$  in [2]. Such a generating function has many desirable properties. It is reparametrization invariant. The integrand is the pullback of the 1-form  $u_\mu(x, \tilde{x})dx^\mu + v_\mu(x, \tilde{x})d\tilde{x}^\mu$  on  $M \times \tilde{M}$ . Since we are interested in theories where the dynamical variables are paths on a manifold we see that (28) is very natural. It is simply the integral of a 1-form along a curve in the product manifold  $M \times \tilde{M}$ . A brief computation shows that

$$\begin{aligned} \pi_\mu(\sigma) &= a_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + b_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{d\tilde{x}^\nu}{d\sigma} , \\ \tilde{\pi}_\mu(\sigma) &= c_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + d_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{d\tilde{x}^\nu}{d\sigma} . \end{aligned}$$

Solving for the domain and range variables we get

$$\frac{d\tilde{x}^\mu}{d\sigma} = A^\mu{}_\nu(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + B^{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \pi_\nu(\sigma) , \quad (29)$$

$$\tilde{\pi}_\mu(\sigma) = C_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + D_\mu{}^\nu(x(\sigma), \tilde{x}(\sigma)) \pi_\nu(\sigma) . \quad (30)$$

This canonical transformation linear in  $dx/d\sigma$  and  $\pi$  is very suggestive of target space duality. The reason is that the hamiltonian is quadratic in  $dx/d\sigma$  and  $\pi$  and thus gets mapped into something which is quadratic in  $d\tilde{x}/d\sigma$  and  $\tilde{\pi}$ . The difficulty is that the new metric and anti-symmetric tensor might not be functions of only  $\tilde{x}$ . *This is the obstruction to duality under ansatz (28)*. Part of our discussion will be to try to understand whether there are local obstructions to the existence of duality transformations. It is worthwhile stating this again explicitly.

Do canonical transformations (29) and (30) lead to a new  $\widetilde{\mathcal{H}}$  such that  $\widetilde{ds}^2$  and  $\widetilde{B}$  are only functions of  $\tilde{x}$ ? In general  $\widetilde{\mathcal{H}}$  will be a non-local function of the variables.

Finally, a more complicated ansatz for (28) which might include terms such as  $d^2x/d\sigma^2$  or  $\sqrt{(dx/d\sigma)^2}$  will generally not lead to transformation equations which are linear in  $dx/d\sigma$  and  $\pi$  as exemplified in (29) and (30).

We proceed to attack these issues. Note that equations (29) and (30), being ordinary differential equations (ODE), are always integrable. To explicitly see this, assume we are given  $(x(\sigma), \pi(\sigma))$  then we insert them into equation (29) which we integrate<sup>1</sup> to obtain  $\tilde{x}(\sigma)$ . Subsequently we insert  $x(\sigma), \pi(\sigma), \tilde{x}(\sigma)$  into (30) to get  $\tilde{\pi}(\sigma)$ . What we are going to do is to replace equations (29) and (30) by an equivalent set of equations (up to ambiguities involving constants of integration).

Introduce new variables  $p$  and  $\tilde{p}$  and define  $p(\sigma)$  and  $\tilde{p}(\sigma)$  by

$$\frac{dp}{d\sigma} = \pi(\sigma), \quad (31)$$

$$\frac{d\tilde{p}}{d\sigma} = \tilde{\pi}(\sigma). \quad (32)$$

It can be shown [14] that the space with coordinates  $(x, p)$  is a manifold  $T^\sharp M$  isomorphic<sup>2</sup> to  $T^*M$ . I do not understand the relationship between the use of the cotangent bundle here and in the interesting work presented in [18, 19]. Canonical transformation equations (29) and (30) may be rewritten as

$$\frac{d\tilde{x}^\mu}{d\sigma} = A^\mu{}_\nu(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + B^{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{dp_\nu}{d\sigma}, \quad (33)$$

$$\frac{d\tilde{p}_\mu}{d\sigma} = C_{\mu\nu}(x(\sigma), \tilde{x}(\sigma)) \frac{dx^\nu}{d\sigma} + D_\mu{}^\nu(x(\sigma), \tilde{x}(\sigma)) \frac{dp_\nu}{d\sigma}. \quad (34)$$

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<sup>1</sup>Of course there are arbitrary constants of integration.

<sup>2</sup>In the case of toroidal target spaces we saw that  $(x, p)$  space was a  $2n$ -torus rather than  $T^*(\mathbf{T}^n)$  due to some global and quantum properties. All our discussions are local thus we will not see such phenomena.

These are integrable because they are ODE's. An advantage of the above over (29) and (30) is that the above treat  $x$  and  $p$  symmetrically.

Is is worthwhile to make a small mathematical digression. Given a manifold  $N$ , let  $PN = \{\gamma : \mathbf{R} \rightarrow N\}$  be the space of all paths in  $N$ . Every map  $f : N \rightarrow \tilde{N}$  induces a map  $f_* : PN \rightarrow P\tilde{N}$ . The opposite is not true. Not all maps from  $PN$  to  $P\tilde{N}$  arise as maps from  $N$  to  $\tilde{N}$ . This is illustrated in equation (33) which is a map from  $P(T^\sharp M)$  to  $P(T^\sharp \tilde{M})$ . Here we explicitly see that the map depends not only on the point  $(x, \tilde{x}) \in T^\sharp M \times T^\sharp \tilde{M}$  but also on the “velocities”  $dx/d\sigma$  and  $dp/d\sigma$ . In general equations (33) and (34) do not define a map from  $T^\sharp M$  to  $T^\sharp \tilde{M}$ . We are interested in looking for a phenomenon similar to that discussed after equation (19) where the map between the phase spaces is induced by a map between  $T^\sharp M$  and  $T^\sharp \tilde{M}$ .

In these lectures we address the question of when do equations (33) and (34) arise as maps from  $T^\sharp M$  to  $T^\sharp \tilde{M}$  in such a way that we get a dual sigma model. We proceed naively by performing a formal mathematical manipulation on (33) and (34): let us multiply both sides of the equations by  $d\sigma$  and obtain the *exterior differential system* (EDS) which will schematically be written as

$$d\tilde{x} = A dx + B dp, \quad (35)$$

$$d\tilde{p} = C dx + D dp. \quad (36)$$

This exterior differentials system is equivalent to the partial differential equations (PDE)  $\partial\tilde{x}/\partial x = A$ ,  $\partial\tilde{x}/\partial p = B$ ,  $\partial\tilde{p}/\partial x = C$  and  $\partial\tilde{p}/\partial p = D$ . In general PDE's have no solutions. There are certain integrability conditions which must be satisfied in order for the system to be integrable. Roughly, we have to be able to integrate simultaneously along  $2n$  independent directions. This is very different from our original ODE system which has no integrability conditions. System (33) and (34) is equivalent to finding one dimensional integrable manifolds of (35) and (36). What we are proposing is that we look for  $2n$ -dimensional integrable manifolds of (35) and (36). If such an integrable manifold exists then we have a map from  $T^\sharp M$  to  $T^\sharp \tilde{M}$ . The integrability conditions for EDS (35) and (36) will describe local geometric conditions which must be satisfied on  $T^\sharp M$  and  $T^\sharp \tilde{M}$  in order for a solution to exist.

## 5.2 Details

We now turn to more detailed study of what was proposed in the last section. Assume we are given a sigma model  $(M, ds^2, B)$ . Let  $\Gamma$  be the Levi-Civita connection associated

with metric  $ds^2$ . We introduce new variables  $p$  and relate them to  $\pi$  via

$$\pi_\mu = \frac{dp_\mu}{d\sigma} - \frac{dx^\lambda}{d\sigma} \Gamma_{\lambda}{}^\nu{}_\mu p_\nu = \frac{\nabla}{d\sigma} p_\mu . \quad (37)$$

This definition is different than the one previously given by equation (18). The definition given above is much better because  $p$  transforms covariantly with respect to coordinate transformations on the base space  $M$ . Just as before one can show that the space with coordinates  $(x, p)$  is a manifold  $T^\sharp M$  isomorphic to the cotangent bundle  $T^*M$ . For any covariant vector  $v_\mu$  define the covariant variation  $\delta_\Gamma v$  by

$$\delta_\Gamma v_\mu = \delta v_\mu - \delta x^\lambda \Gamma_{\lambda}{}^\nu{}_\mu v_\nu . \quad (38)$$

A brief computation shows that

$$\delta_\Gamma \pi_\mu = \frac{\nabla}{d\sigma} \delta_\Gamma p_\mu - R^\nu{}_{\mu\lambda\rho} \delta x^\lambda \frac{dx^\rho}{d\sigma} p_\nu . \quad (39)$$

The symplectic structure (3) may be rewritten as

$$\begin{aligned} & \int d\sigma \delta x^\mu \wedge \delta \pi_\mu \\ &= \frac{1}{2} \int d\sigma \begin{pmatrix} \delta x & \delta_\Gamma p \end{pmatrix} \wedge \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \nabla/d\sigma & 0 \\ b & \nabla/d\sigma \end{pmatrix} \begin{pmatrix} \delta x \\ \delta_\Gamma p \end{pmatrix} , \end{aligned} \quad (40)$$

where  $b_{\mu\nu} = p_\lambda (dx^\rho/d\sigma) R^\lambda{}_{\rho\mu\nu}$ . Note that our old friend  $Q$  appears in the above. What is surprising is that

$$\begin{pmatrix} \nabla/d\sigma & 0 \\ b & \nabla/d\sigma \end{pmatrix} \quad (41)$$

is the unique torsion free pseudo-riemannian connection on  $T^\sharp M$  associated with the  $Q$ -metric

$$ds_Q^2 = dx^\mu \otimes (dp_\mu - dx^\lambda \Gamma_{\lambda}{}^\nu{}_\mu p_\nu) + (dp_\mu - dx^\lambda \Gamma_{\lambda}{}^\nu{}_\mu p_\nu) \otimes dx^\mu \quad (42)$$

on  $T^\sharp M$ . A short computation shows that (41) is a skew-adjoint operator with respect to the  $Q$ -metric.

We see that some natural geometric structures related to the pseudo-riemannian geometry of the  $Q$ -metric are beginning to appear.

### 5.3 A digression and an analogous problem

We expand on the analogy discussed in Table 1. Assume we are in euclidean space  $\mathbf{R}^n$ . The euclidean group  $E(n)$  is the group of isometries of euclidean space. Since Felix

Klein we have understood that euclidean geometry is the study of the properties of figures which are invariant under the action of the euclidean group (the isometry group of euclidean space). In a similar fashion one can define hyperbolic and elliptic geometries. Klein's ideas seem to fail when one considers riemannian geometry: on a generic riemannian manifold  $M$  there are no isometries. E. Cartan realized that this was not fatal and that the orthogonal group played a very important role in riemannian geometry. Cartan reformulated euclidean geometry using the observation that  $\mathbf{R}^n = E(n)/O(n)$ . In modern language,  $E(n)$  is a principal  $O(n)$ -bundle over  $\mathbf{R}^n$ . Cartan studied the properties of  $\mathbf{R}^n$  in terms of the properties of  $E(n)$ . In doing so he realized that the Maurer-Cartan equations for the group  $E(n)$  contain all the information necessary to extract both the properties of  $\mathbf{R}^n$  and  $O(n)$ . Cartan now attacked the problem of riemannian geometry by observing that on every riemannian manifold  $(M, ds^2)$  one could construct a larger space  $O(M)$  called the bundle of orthonormal frames. For a euclidean space, the bundle of orthonormal frames may be identified with  $E(n)$ . Cartan was able to write down a generalization of the Maurer-Cartan equations on the bundle  $O(M)$ . These equations are known as the first and second structural equations of the space. What Cartan discovered was that in riemannian geometry there was a group action of  $O(n)$  on the bundle of frames  $O(M)$  rather than on the base space  $M$ . The base space  $M = O(M)/O(n)$  and all properties of  $M$  may be understood in terms of the properties of  $O(M)$ . This is the starting point for modern riemannian geometry.

Assume we have an isometry from a riemannian manifold  $(M, ds^2)$  to a riemannian manifold  $(\tilde{M}, \tilde{ds}^2)$ . This leads to a system of non-linear PDE's given by

$$\tilde{g}_{\mu\nu}(\tilde{x}) \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} = g_{\rho\sigma}(x) . \quad (43)$$

Cartan realized that there was a better and more geometric way to formulate these equations. Cartan constructed local orthonormal coframes  $\omega^m = e^m_\mu dx^\mu$  on  $M$ , and  $\tilde{\omega}$  on  $\tilde{M}$  where the metrics may be written as  $ds^2 = \omega^m \otimes \omega^m$  and  $\tilde{ds}^2 = \tilde{\omega}^m \otimes \tilde{\omega}^m$ . An isometry requires the existence of an orthogonal matrix-valued function  $R$  such that  $\tilde{\omega}^m = R^m_n \omega^n$ . Cartan now observed that this leads us to a first order EDS  $\tilde{e}^m_\mu d\tilde{x}^\mu = R^m_n e^n_\nu dx^\nu$ . Cartan went much further. He realized that one could promote  $R$  to a new independent variable. This is similar to introducing a new variable  $u = dy/dx$  in a second order ODE and writing the original equation as a pair of first order ODEs. Instead of working on  $M$ , Cartan worked on a space with coordinates  $(x, R)$ . In modern language, this is the bundle of orthonormal frames. Also, the structural equations are globally defined on the bundle of orthonormal frames whereas (43) are only valid locally. This is what led to his invention of the theory of  $G$ -structures (principal sub-bundles of the frame bundle) and generalized geometries.

In the modern viewpoint, an isometry may be defined in the following way. Assume we have a map  $f : M \rightarrow \widetilde{M}$ ; the differential of this map  $df : TM \rightarrow T\widetilde{M}$  naturally acts on the bundle of *all* frames, *i.e.*,  $\frac{\partial}{\partial x^\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \frac{\partial}{\partial \tilde{x}^\mu}$ . If  $df$  lifts to a bundle map from the *orthonormal frame bundle*  $O(M)$  to the *orthonormal frame bundle*  $O(\widetilde{M})$  then  $f$  is an isometry. We have to be careful with the converse. A bundle map  $O(M) \rightarrow O(\widetilde{M})$  is fiber-preserving and thus induces a map  $M \rightarrow \widetilde{M}$  on the base spaces. In general, the bundle map from  $O(M)$  to  $O(\widetilde{M})$  will not be an isometry because it will not be the lift of a map between the bases.

When is a bundle map  $F : O(M) \rightarrow O(\widetilde{M})$  the lift of a map  $f : M \rightarrow \widetilde{M}$ ? Recall that frame bundles are endowed with a globally defined  $\mathbf{R}^n$ -valued canonical 1-form. Let  $\theta$  and  $\tilde{\theta}$  be the respective canonical 1-forms on  $O(M)$  and  $O(\widetilde{M})$ . If  $F^*\tilde{\theta} = \theta$  then  $F$  is the lift of a map between the bases. This means that an isometry can be defined as a bundle map between orthogonal frame bundles which preserves the canonical 1-forms.

The formulation just presented is global whereas the formulation in terms of PDE's (43) is local. The question of the existence of an isometry between riemannian manifolds is a difficult one. There are both local and global issues which must be addressed. In this report we only discuss local issues. Roughly, the local existence of an isometry is guaranteed if there exists coordinate systems on  $M$  and  $\widetilde{M}$  such that the curvature and its higher derivatives agree up to a certain order. Global issues are much more difficult [20].

The key point is contained in the following scenario.

Assume we have a pseudo-isometry from  $(T^\sharp M, ds_Q^2)$  to  $(T^\sharp M, \tilde{ds}_Q^2)$ . It is relatively easy to show that not only does the pseudo-isometry preserve the metric but also the connection. Consequently, the induced map on paths preserves the symplectic form (40). This means that the pseudo-isometry between the finite dimensional spaces  $T^\sharp M$  and  $T^\sharp \widetilde{M}$  induces a canonical transformation from the infinite dimensional phase space  $(x(\sigma), \pi(\sigma))$  to the infinite dimensional phase space  $(\tilde{x}(\sigma), \tilde{\pi}(\sigma))$ .

The above requires some qualifications because we have ignored global issues<sup>3</sup>. The construction just described justifies the analogy presented in Table 1.

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<sup>3</sup>A discussion of global issues in duality may be found in [21].

## 5.4 Back to duality

We will now perform the mathematical construction we require. We begin with a sigma model  $(M, ds^2, B)$ . Using the metric we construct a local orthonormal coframe  $\{\omega^m\}$ . We put coordinates on the “fake cotangent bundle”  $T^\sharp M$  by noting that a 1-form may be written as  $p_m \omega^m$ . This allows us to define 1-forms  $\mu_m = dp_m - \omega^l \Gamma_{lm}^n p_n$ . These 1-forms define the horizontal distribution on  $T^\sharp M$  associated with the Levi-Civita connection. The  $Q$ -metric on  $T^\sharp M$  is given by

$$ds_Q^2 = \omega^m \otimes \mu_m + \mu_m \otimes \omega^m . \quad (44)$$

A pseudo-isometry is a map from  $T^\sharp M$  to  $T^\sharp \widetilde{M}$  such that

$$\begin{pmatrix} \tilde{\omega} \\ \tilde{\mu} \end{pmatrix} = S \begin{pmatrix} \omega \\ \mu \end{pmatrix} , \quad (45)$$

where  $S \in O_Q(2n)$ . We now rephrase the above in the language of bundles.

Given the metric  $ds_Q^2$  on  $T^\sharp M$  we construct the bundle of  $Q$ -orthonormal frames, denoted by  $O_Q(T^\sharp M)$ . This bundle has base space  $T^\sharp M$  and fiber  $O_Q(2n)$ . A pseudo-isometry between  $(T^\sharp M, ds_Q^2)$  and  $(T^\sharp \widetilde{M}, \widetilde{ds}_Q^2)$  is given by a bundle map between  $O_Q(T^\sharp M)$  and  $O_Q(T^\sharp \widetilde{M})$  which preserves the canonical 1-forms. Once we have a pseudo-isometry we can construct the desired canonical transformation. This is the full story as far as the “canonical structure” is concerned. This is only half the problem because we have to worry about the hamiltonian.

## 6 The hamiltonian structure

The hamiltonian is related to a positive definite metric on  $T^\sharp M$  defined by

$$ds_{\mathcal{H}}^2 = (\mu_m - B_{mn} \omega^n) \otimes (\mu_m - B_{ml} \omega^l) + \omega^m \otimes \omega^m . \quad (46)$$

Note that for any curve  $(x(\sigma), p(\sigma))$  on  $T^\sharp M$ , the evaluation of the above on the tangent vector to the curve yields  $2\mathcal{H}$  where  $\mathcal{H}$  is defined by (5).

We are now in a position to see what is required to have duality within our scenario. Assume we have a  $Q$ -pseudo-isometry  $f : T^\sharp M \rightarrow T^\sharp \widetilde{M}$ . If in addition,  $f$  preserves the metric  $ds_{\mathcal{H}}^2$  then the sigma model  $(M, ds^2, B)$  will be dual to the sigma model  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$ . We are interested in maps from  $T^\sharp M$  to  $T^\sharp \widetilde{M}$  which preserve two different symmetric 2-tensors,  $ds_Q^2$  and  $ds_{\mathcal{H}}^2$ , of different type. We will call such maps



$K$ -isometries. To formulate this problem in terms of bundles we have to use the bundle consisting of frames which are simultaneously orthonormal with respect to  $ds_Q^2$  and  $ds_{\mathcal{H}}^2$ . The fiber of this bundle is isomorphic to  $K = O_Q(2n) \cap O(2n) \approx O(n) \times O(n)$ , the maximal compact subgroup of  $O_Q(2n)$ . We can now state the main result of these lectures. Let  $K(T^\sharp M, ds_Q^2, ds_{\mathcal{H}}^2)$  be the bundle consisting of frames which are simultaneously orthonormal with respect to  $ds_Q^2$  and  $ds_{\mathcal{H}}^2$ .

**Theorem 47** *Assume there exists a bundle map between the frame bundles  $K(T^\sharp M, ds_Q^2, ds_{\mathcal{H}}^2)$  and  $K(T^\sharp \widetilde{M}, \widetilde{ds}_Q^2, \widetilde{ds}_{\mathcal{H}}^2)$  which preserves the canonical 1-forms; then the sigma model  $(M, ds^2, B)$  is dual to the sigma model  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$ . Said differently, the sigma model  $(M, ds^2, B)$  is dual to the sigma model  $(\widetilde{M}, \widetilde{ds}^2, \widetilde{B})$  if there exists a  $K$ -isometry between  $T^\sharp M$  and  $T^\sharp \widetilde{M}$ .*

To relate these ideas to concepts familiar from toroidal target space duality we discuss in more detail the bundle  $K(T^\sharp M, ds_Q^2, ds_{\mathcal{H}}^2)$ . A general  $Q$ -orthonormal frame at a point in  $T^\sharp M$  may be written as

$$S \begin{pmatrix} \omega \\ \mu \end{pmatrix} \quad (48)$$

where  $S \in O_Q(2n)$ . First we decompose  $S$  in a way that explicitly exhibits the hamiltonian. The Lie algebra of  $O_Q(2n)$  may be written in a block decomposition as

$$\begin{pmatrix} \delta & \beta \\ \gamma & -\delta^t \end{pmatrix}, \quad (49)$$

each entry is an  $n \times n$  matrix and  $\beta, \gamma$  are skew. A matrix in the Lie algebra of  $K$  is of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad (50)$$

where  $\alpha, \beta$  are skew. This suggests a decomposition

$$\begin{pmatrix} \delta & \beta \\ \gamma & -\delta^t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}, \quad (51)$$

where  $\sigma$  is symmetric and  $\epsilon$  is skew. Near the identity one can rewrite (48) in the form

$$T \begin{pmatrix} e^\sigma & 0 \\ 0 & e^{-\sigma} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} \omega \\ \mu \end{pmatrix} = T \begin{pmatrix} e^{+\sigma} \omega \\ e^{-\sigma} (\mu - B\omega) \end{pmatrix}, \quad (52)$$

where  $T \in K$  and  $B$  is skew. Note that  $e^{\pm\sigma}$  are non-singular symmetric matrices. Since  $T$  is orthogonal, we see that we have essentially computed a square root for (5).

We have explicitly found the  $B$  field and the  $e^\sigma$  term is roughly the square root of  $ds^2$ . This shows that at a point on  $T^\sharp M$  the moduli space for  $ds_{\mathcal{H}}^2$  is given by  $O_Q(2n)/K$ . The moduli space depends on  $n^2$  real parameters. The reader should compare this with the standard discussion in toroidal target spaces.

It is now worthwhile to consider a second decomposition which is reminiscent of both the Iwasawa decomposition and the Euler angle decomposition. Observe that there exists an  $R \in O(n)$  such that  $\sigma = R^{-1}\Delta R$  where  $\Delta$  is diagonal. What we do is insert this into (52), absorb an  $R^{-1}$  into  $T$ , redefine  $B$  and push  $R$  all the way to the right.

$$T' \begin{pmatrix} e^{+\Delta} & 0 \\ 0 & e^{-\Delta} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B' & I \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \omega \\ \mu \end{pmatrix} = T' \begin{pmatrix} e^{+\Delta}\omega' \\ e^{-\Delta}(\mu' - B'\omega') \end{pmatrix}. \quad (53)$$

In the above  $T' \in K$ ,  $B'$  is skew,  $\omega' = R\omega$  and  $\mu' = R\mu$ . Many explicit examples of target space duality directly give a hamiltonian which can be put into this final form. This decomposition depends crucially on the details of how the subgroup  $K$  sits inside of  $O_Q(2n)$ .

The symplectic structure on the infinite dimensional phase space is associated with  $ds_Q^2$ . This gives us a frame bundle with fiber  $O_Q(2n)$  as in Figure 2. Fix a point in  $T^\sharp M$ . Each orbit of  $K$  in  $O_Q(2n)$  determines a hamiltonian at that point. A choice of a  $K$ -orbit on the fiber over each point in  $T^\sharp M$  is same as a choice of a sub-bundle with reduced structure group  $K$ . In general this bundle will not be associated with a hamiltonian. Remember that earlier we were careful to state that the moduli space for the hamiltonian *at a point* in  $T^\sharp M$  was  $O_Q(2n)/K$ . If  $(x, p)$  are local coordinates on  $T^\sharp M$  then observe that  $g_{\mu\nu}(x)$  and  $B_{\mu\nu}(x)$  are only functions of  $x$ . They do not depend on  $p$ . Bundles of type  $K(T^\sharp M, ds_Q^2, ds_{\mathcal{H}}^2)$  are a subset of the set of bundles of  $K$ -orthonormal frames. This has important consequences when studying the principal  $K$ -bundles over  $T^\sharp M$  because the bundles  $K(T^\sharp M, ds_Q^2, ds_{\mathcal{H}}^2)$  are very special. If one writes the structural equations for a general principal  $K$ -bundle one finds many curvature and torsion terms. The fact that  $ds^2$  and  $B$  “come from  $M$ ” imposes many constraints on the structural equations which are discussed in [14]. The structural equations contain the information necessary to understand the local obstructions to the existence of  $K$ -isometries.

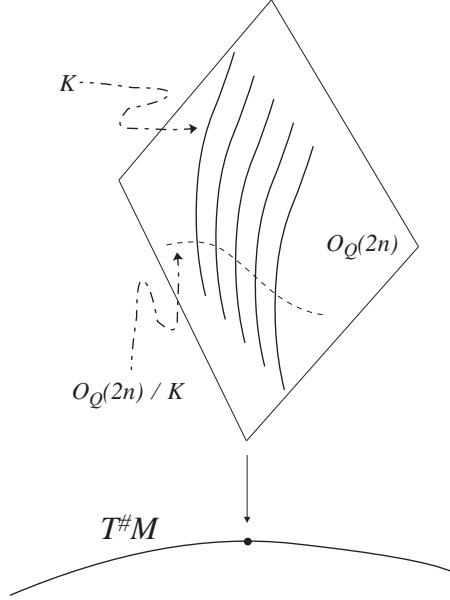


Figure 2: The quadrilateral represents the fiber  $O_Q(2n)$  over a point in  $T^\#M$ . The coset space  $O_Q(2n)/K$  arises from a fibration of  $O_Q(2n)$  with the fibers isomorphic to  $K$ . The leaves of the fibration are depicted by the heavy vertical curves. We schematically denote the orbit space  $O_Q(2n)/K$  by the dashed curve.

## 7 A challenge to mathematicians

The content of these lectures suggest a very interesting mathematical problem. Can one find examples of geometrical data which would lead to nontrivial *local*  $K$ -isometries between  $T^\#M$  and  $T^\#\widetilde{M}$ ? The reason for the requirement of locality is that in these lectures we did not discuss global issues except in the case of toroidal target spaces. The existence of local isomorphisms between the principal bundles  $K(T^\#M, ds_Q^2, ds_{\mathcal{H}}^2)$  and  $K(T^\#\widetilde{M}, \widetilde{ds}_Q^2, \widetilde{ds}_{\mathcal{H}}^2)$  which preserve the canonical 1-forms is intimately related to Cartan's problem of equivalence [13]. There one finds that the existence of such local isomorphisms is codified in the curvature and torsion of the bundles. Explicit formulas will appear in [14]. The only explicit examples known of this phenomenon are the toroidal target spaces. It is not clear whether the non-abelian duality examples fall in this “restricted” class. Global existence theorems are much more complicated [20].

There are well established procedures for constructing the dual theory when the sigma model admits a continuous group action [5, 6, 8, 9, 11]. Is there a systematic procedure which can be invoked to look for the existence of dual target spaces when there are no symmetries?

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